

ON INVARIANT GRAPH SUBSPACES OF A J -SELF-ADJOINT OPERATOR IN THE FESHBACH CASE*

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We consider a J -self-adjoint 2×2 block operator matrix L in the Feshbach spectral case, that is, in the case where the spectrum of one main-diagonal entry is embedded into the absolutely continuous spectrum of the other main-diagonal entry. We work with the analytic continuation of the Schur complement of a main-diagonal entry in $L - z$ to the unphysical sheets of the spectral parameter z plane. We present the conditions under which the continued Schur complement has operator roots, in the sense of Markus-Matsaev. The operator roots reproduce (parts of) the spectrum of the Schur complement, including the resonances. We then discuss the case where there are no resonances and the associated Riccati equations have bounded solutions allowing the graph representations for the corresponding J -orthogonal invariant subspaces of L . The presentation ends with an explicitly solvable example.

1. INTRODUCTION

In this note we deal with a 2×2 block operator matrix of the form

$$L = \begin{pmatrix} A_0 & B \\ -B^* & A_1 \end{pmatrix} \quad (1.1)$$

It is assumed that A_0 and A_1 are self-adjoint operators in Hilbert spaces \mathfrak{A}_0 and \mathfrak{A}_1 , respectively, and B is a bounded operator from \mathfrak{A}_1 to \mathfrak{A}_0 . The block operator matrix L is understood as an operator in the orthogonal sum $\mathfrak{H} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ of the Hilbert spaces \mathfrak{A}_0 and \mathfrak{A}_1 , and $\text{Dom}(L) = \text{Dom}(A_0) \oplus \text{Dom}(A_1)$.

When studying operators of the form (1.1), one usually introduces the involution

$$J = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (1.2)$$

where I denotes the identity operator. In this case the product JL is a self-adjoint operator in \mathfrak{H} , and then the operator L is called J -self-adjoint. Surely, the J -self-adjoint operator (1.1) may be viewed as a perturbation, $L = A + V$, of the block-diagonal self-adjoint operator matrix

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$A := \text{diag}(A_0, A_1)$, $\text{Dom}(A) = \text{Dom}(A_0) \oplus \text{Dom}(A_1)$, by the bounded J -self-adjoint off-diagonal operator block matrix

$$V = \begin{pmatrix} 0 & B \\ -B^* & 0 \end{pmatrix}. \quad (1.3)$$

The involution (1.2) induces an indefinite inner product

$$[x, y] = (Jx, y), \quad x, y \in \mathfrak{H}. \quad (1.4)$$

Equipping the Hilbert space \mathfrak{H} with the indefinite inner product (1.4) transforms it into a Krein space, which we denote by \mathfrak{K} , $\mathfrak{K} = \{\mathfrak{H}, [\cdot, \cdot]\}$. Notice that if an operator is J -self-adjoint in \mathfrak{H} then it is self-adjoint in \mathfrak{K} . In particular, the operator (1.1) is self-adjoint in \mathfrak{K} . The theory of linear and, in particular, self-adjoint operators in Krein spaces is already a deeply developed subject and for the corresponding definitions, concepts and main results we refer the reader, e.g., to [14], [9], or [7]. For the recent spectral results and further references see, e.g., [6] and [20]

Surely, for $B \neq 0$ the J -self-adjoint operator L given by (1.1) can not be self-adjoint in \mathfrak{H} with respect to the original inner product. Nevertheless, in many cases the spectrum of such an operator is purely real and, moreover, L turns out to be similar to a self-adjoint operator. Such a situation takes place if L possesses a couple of complementary J -orthogonal reducing subspaces \mathfrak{L}_0 and \mathfrak{L}_1 that are maximal uniformly definite (respectively, positive and negative) with respect to the Krein space inner product (1.4) (see, e.g., the papers [4, 5] and references cited therein). Basically, this happens for the case where the spectra

$$\sigma_0 := \text{spec}(A_0) \quad \text{and} \quad \sigma_1 := \text{spec}(A_1) \quad (1.5)$$

of the entries A_0 and A_1 are disjoint, i.e.,

$$\delta := \text{dist}(\sigma_0, \sigma_1) > 0 \quad (1.6)$$

and the norm of B is sufficiently small (see [4, Theorem 5.8] or [5, Theorem 6.1]): In general, we need to have $\|V\| < \frac{\delta}{\pi}$ but if the spectral sets σ_0 and σ_1 are separated from each other by only one or two gaps, then the sufficient condition reduces to the bound $\|V\| < \frac{\delta}{2}$. Sufficient conditions for the similarity of a J -self-adjoint operator to a self-adjoint one are also known in the case of some unbounded B [24, 25].

The maximal uniform definiteness of the subspaces \mathfrak{L}_0 and \mathfrak{L}_1 suggests the existence of strictly contractive operators $Y \in \mathcal{B}(\mathfrak{A}_1, \mathfrak{A}_0)$ and $\widehat{Y} = Y^*$ (see, e.g., [5, Section 2]) such that \mathfrak{L}_0 is the graph of \widehat{Y} and \mathfrak{L}_1 is the graph of Y ,

$$\mathfrak{L}_0 = \mathcal{G}(\widehat{Y}) := \{x_0 \oplus \widehat{Y}x_0 \mid x_0 \in \mathfrak{A}_0\}, \quad \mathfrak{L}_1 = \mathcal{G}(Y) := \{Yx_1 \oplus x_1 \mid x_1 \in \mathfrak{A}_1\}. \quad (1.7)$$

The angular operators Y and \widehat{Y} are strong solutions for the pair of respective dual operator Riccati equations:

$$A_0Y - YA_1 + YB^*Y = -B. \quad (1.8)$$

and

$$\widehat{Y}A_0 - A_1\widehat{Y} + \widehat{Y}B\widehat{Y} = -B^*. \quad (1.9)$$

In the present work we are concerned with the case where condition (1.6) fails to hold from the very beginning: We assume that the entry A_0 has only absolutely continuous spectrum and that the spectrum of A_1 is completely embedded into the spectrum of A_0 , that is,

$$\sigma_1 \subset \sigma_0. \quad (1.10)$$

For the case (1.10) one knows that, in general, the operator L has complex spectrum and that the following enclosure holds: $\text{spec}(L) \setminus \mathbb{R} \subset \{z \in \mathbb{C} \mid |\text{Im} z| \leq \|B\|\}$ (see [23, Theorem 5.5]). It is also known that if A is bounded or semibounded then $\inf \text{spec}(A) \leq \text{Re spec}(L) \leq \sup \text{spec}(A)$ (see [5, Theorem 5.8]).

In order to study the spectral problem for the block operator matrix (1.1), we employ the Frobenius-Schur factorization (see. e.g., [22, Proposition 1.6.2 and Theorem 2.2.13]) of the difference $L - z$, $z \notin \sigma_0$:

$$L - z = \begin{pmatrix} I & 0 \\ -B^*(A_0 - z)^{-1} & I \end{pmatrix} \begin{pmatrix} A_0 - z & 0 \\ 0 & M_1(z) \end{pmatrix} \begin{pmatrix} I & (A_0 - z)^{-1}B \\ 0 & I \end{pmatrix}, \quad (1.11)$$

where $M_1(z)$ stands for the Schur complement of $A_0 - z$,

$$M_1(z) = A_1 - z + W_1(z) \quad (1.12)$$

with

$$W_1(z) = B^*(A_0 - z)^{-1}B, \quad (1.13)$$

Notice that the resolvent $(L - z)^{-1}$ can be expressed explicitly in terms of the inverse $M_1^{-1}(z)$; (1.11) also implies that $\text{spec}(L) \setminus \text{spec}(A_0) \subseteq \text{spec}(M_1)$. Therefore, in studying the spectral properties of the transfer function M_1 one studies at the same time the spectral properties of the operator matrix L .

Assuming that the absolutely continuous spectrum σ_0 consist of the only branch presented by a finite or infinite closed interval $\Delta_0 \subset \mathbb{R}$, in Section 2 we formulate conditions on B allowing to perform analytic continuation of the Schur complement $M_1(z)$ through the cut along Δ_0 to certain domains lying on the neighboring unphysical sheets of the spectral parameter plane. Here we follow exactly the line of the work [16] in its simplified version [17].

Having two variants of the continued Schur complement M_1 , produced by crossing the cut Δ_0 from \mathbb{C}^+ down and from \mathbb{C}^- up, for both of them in Section 3 we prove the existence of the respective operator roots $Z^{(l)}$, $l = -1$ and $l = +1$. The spectrum of the operator root $Z^{(l)}$, $l = \pm 1$, is just the spectrum of the analytically continued Schur complement M_1 lying at some neighborhood of the set σ_1 . The size of this neighborhood is determined by the strength of the operator B . The spectrum of $Z^{(l)}$ along with (a part of) the spectrum of L may include resonances (by which we understand the complex spectrum of M_1 located in the continuation domain on the corresponding unphysical sheet).

In Section 4 we discuss the case where the operator $Z^{(l)}$, $l = \pm 1$, has no real and resonant spectrum. In this case, under minimal additional assumptions, the operator Riccati equations (1.8) and (1.9) are proven to be solvable. However, unlike in the cases of disjoint spectral components σ_0 and σ_1 considered in [4, 5], now the operator L has complex spectrum and the L -invariant graph subspaces (1.7) are not maximal uniformly definite.

Section 5 presents an example that just fits the main assumptions of Section 4. Namely, in Section 5 we deal with the spectral disposition that is called Feshbach — in complete analogy with the celebrated similar one in the case of quantum-mechanical Hamiltonians. We assume that the subspace \mathfrak{A}_1 is finite-dimensional and that the perturbation B is such that it completely sweeps the eigenvalues of A_1 (which are all embedded into the absolutely continuous spectrum of A_0) from the real axis. However, just opposite to the Hermitian case, the eigenvalues of A_1 turn not into resonances but into the complex spectrum of L . The resulting operator roots $Z^{(l)}$,

$l = \pm 1$, have neither real no resonance spectrum and, thus, J -self-adjoint block operator matrix (1.1) possesses a couple of mutually J -orthogonal invariant graph subspaces of the form (1.7).

Finally, in Section 6 we present the simplest possible example with A_0 being the operator of multiplication by independent variable in $\mathfrak{A}_0 = L_2(-\alpha, \alpha)$, $\alpha > 0$, and $A_1 = a_1$ being the multiplication by a number $a_1 \in (-\alpha, \alpha)$ in $\mathfrak{A}_1 = \mathbb{C}$. At least for $a_1 = 0$, the norm of the corresponding solutions Y and $\hat{Y} = Y^*$ in this example is computed explicitly: $\|Y\| = \|\hat{Y}\| = 1$.

The following notations are used throughout the paper. By \mathbb{C}^+ and \mathbb{C}^- we understand respectively the upper and lower half-planes of the complex plane \mathbb{C} , e.g., $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im } z > 0\}$. By a subspace of a Hilbert space we always mean a closed linear subset. The Banach space of bounded linear operators from a Hilbert space \mathfrak{M} to a Hilbert space \mathfrak{N} is denoted by $\mathcal{B}(\mathfrak{M}, \mathfrak{N})$ and by $\mathcal{B}(\mathfrak{M})$ if $\mathfrak{N} = \mathfrak{M}$. The notation $E_T(\sigma)$ is always used for the spectral projection of a self-adjoint operator T associated with a Borel set $\sigma \subset \mathbb{R}$. By $\mathcal{O}_r(\sigma)$, $r \geq 0$, we denote the closed r -neighbourhood of σ in \mathbb{C} , i.e. $\mathcal{O}_r(\sigma) = \{z \in \mathbb{C} \mid \text{dist}(z, \sigma) \leq r\}$. We let $\text{Dom}(Z)$ and $\text{Ran}(Z)$ denote the respective domain and range of a linear operator Z .

2. ANALYTIC CONTINUATION OF THE SCHUR COMPLEMENT

In this note we restrict our consideration to the case where all the spectrum $\sigma_0 := \text{spec}(A_0)$ of the entry A_0 is absolutely continuous and coincides with the closure of the single interval $\Delta_0 := (\mu_0^{(1)}, \mu_0^{(2)})$ with $-\infty \leq \mu_0^{(1)} < \mu_0^{(2)} \leq \infty$. Furthermore, the whole spectrum $\sigma_1 := \text{spec}(A_1)$ of the entry A_1 is assumed to be embedded into the interval Δ_0 , that is, $\sigma_1 \subset \Delta_0$.

Denote by E_0 the spectral measure of the self-adjoint operator A_0 and let $E^0(\mu) := E_0((-\infty, \mu))$ be the spectral function of A_0 . Then the function $W_1(z)$ can be written as

$$W_1(z) = \int_{\sigma_0} dK_B(\mu)(\mu - z)^{-1} \quad (2.1)$$

where

$$K_B(\mu) = B^* E^0(\mu) B. \quad (2.2)$$

Our central assumption is that the operator-valued function $K_B(\mu)$ is differentiable in $\mu \in \Delta_0$ in the operator norm topology and that the derivative $K'_B(\mu)$ admits analytic continuation from Δ_0 to a simply connected domain D^- located in \mathbb{C}^- . We suppose that the boundary of D^- includes the entire spectral interval Δ_0 . Since $K'_B(\mu)$ represents a self-adjoint operator for any $\mu \in \Delta_0$ and $\Delta_0 \subset \mathbb{R}$, the function $K'_B(\mu)$ also admits analytic continuation from Δ_0 into the domain D^+ , symmetric to D^- with respect to the real axis and $[K'_B(\mu)]^* = K'_B(\bar{\mu})$, $\mu \in D^\pm$. For the case where the end point $\mu_0^{(k)}$, $k = 1, 2$, of the spectral interval Δ_0 is finite we will always suppose that

$$\|K'_B(\mu)\| \leq C |\mu - \mu_0^{(k)}|^\gamma, \quad \mu \in D_k^\pm, \quad (2.3)$$

with some $C > 0$ and $\gamma \in (-1, 0]$.

In the notations like D^+ and D^- below we will often use the number index l , $l = +1$ or $l = -1$, identifying the values of l in the notation D^l with the respective signs “+” or “-”.

Let Γ^l , $l = \pm 1$, be a rectifiable Jordan curve originating from a continuous deformation of the interval Δ_0 and lying in D^l , with the (finite) end points fixed. The quantity

$$\mathcal{V}_0(B, \Gamma^l) := \int_{\Gamma^l} |d\mu| \|K'_B(\mu)\| \quad (2.4)$$

where $|d\mu|$ stands for the Lebesgue measure on the contour Γ^l , is called the variation of the operator-valued function $K_B(\mu)$ along Γ^l . We suppose that there are contours Γ^l with finite $\mathcal{V}_0(B, \Gamma^l)$ even in the case of unbounded Δ_0 . Jordan contours Γ^l satisfying condition $\mathcal{V}_0(B, \Gamma^l) < \infty$ are said to be *admissible*.

We will need the following elementary statement (cf., e.g., [16, Lemma 2.1]).

Lemma 2.1. *The analytic continuation of the Schur complement $M_1(z)$, $z \in \mathbb{C} \setminus \sigma_0$, through the spectral interval Δ_0 into the subdomain $D(\Gamma^l) \subset D^l$, $l = \pm 1$, bounded by Δ_0 and an admissible contour Γ^l is given by*

$$M_1(z, \Gamma^l) = A_1 - z + W_1(z, \Gamma^l) \quad \text{with} \quad W_1(z, \Gamma^l) = \int_{\Gamma^l} d\mu K'_B(\mu) (\mu - z)^{-1}. \quad (2.5)$$

For $z \in D^l \cap D(\Gamma^l)$ one has

$$M_1(z, \Gamma^l) = M_1(z) - 2\pi i l K'_B(z). \quad (2.6)$$

Proof. The proof is reduced to the observation that the function $M_1(z, \Gamma^l)$ is holomorphic for $z \in \mathbb{C} \setminus \Gamma^l$ and coincides with $M_1(z)$ for $z \in \mathbb{C} \setminus \overline{D(\Gamma^l)}$. The last equation representing $M_1(z, \Gamma^l)$ via $M_1(z)$ is obtained from (2.5) by using the Residue Theorem. \square

Remark 2.2. From the representation (2.6) it follows that the Riemann surface of the Schur complement $M_1(z)$ is larger than the single sheet of the spectral parameter plane. The sheet of the complex plane \mathbb{C} where the function $M_1(z)$ and the resolvent $(L - z)^{-1}$ are considered initially is called the physical sheet. Formula (2.6) implies that the domains D^- and D^+ are to be placed on additional sheets of the z plane that are different from the physical sheet. We remind that these additional sheets are usually called unphysical sheets (see, e.g., [21]). In this work we only deal with the unphysical sheets attached (through the interval Δ_0) immediately to the physical sheet.

3. A FACTORIZATION RESULT

Suppose that the spectrum of a linear operator $Z \in \mathcal{B}(\mathfrak{A}_1)$ does not intersect an admissible contour $\Gamma \subset D^\pm$. Then one can introduce the (transformator) operator

$$W_1(Z, \Gamma) = - \int_{\Gamma} d\mu K'_B(\mu) (Z - \mu)^{-1}. \quad (3.1)$$

Clearly, if the resolvent $(Z - \mu)^{-1}$, $\mu \in \Gamma$, is uniformly bounded on Γ , the operator $W_1(Z, \Gamma)$ is also bounded,

$$\|W_1(Z, \Gamma)\| \leq \mathcal{V}_0(B, \Gamma) \sup_{\mu \in \Gamma} \|(Z - \mu)^{-1}\|. \quad (3.2)$$

Below, we consider the *basic* equation (cf. [19])

$$Z = A_1 + W_1(Z, \Gamma) \quad (3.3)$$

or, equivalently,

$$X = W_1(A_1 + X, \Gamma). \quad (3.4)$$

assuming that a solution X of the latter is looked for in $\mathcal{B}(\mathfrak{A}_1)$.

It is worth noting that if X is a solution of (3.4) and u_1 is an eigenvector of $Z = A_1 + X$ corresponding to an eigenvalue z , $Zu_1 = zu_1$, then by (3.1) and (3.3) we have

$$zu_1 = A_1 u_1 + W_1(Z, \Gamma) u_1 = A_1 u_1 + W_1(z, \Gamma) u_1. \quad (3.5)$$

Thus, any eigenvalue z of Z is simultaneously an eigenvalue for the continued transfer function $M_1(z, \Gamma)$ while u_1 is an eigenvector of this function.

The next statement is a variant of an existence result from [16, Theorem 3.1] (cf. similar results in [11, 12]) rewritten for the J -self-adjoint case under consideration.

Theorem 3.1. *Assume that Γ is an admissible contour satisfying the condition*

$$\mathcal{V}_0(B, \Gamma) < \frac{1}{4} d^2(\Gamma), \quad (3.6)$$

where $d(\Gamma) := \text{dist}(\sigma_1, \Gamma)$. Then equation (3.4) has a unique solution in any closed ball in $\mathcal{B}(\mathfrak{A}_1)$ consisting of operators X satisfying $\|X\| \leq r$ with $r_{\min}(\Gamma) \leq r < r_{\max}(\Gamma)$ where

$$r_{\min}(\Gamma) = d(\Gamma)/2 - \sqrt{d^2(\Gamma)/4 - \mathcal{V}_0(B, \Gamma)}, \quad r_{\max}(\Gamma) = d(\Gamma) - \sqrt{\mathcal{V}_0(B, \Gamma)}. \quad (3.7)$$

The unique solution X belongs to the smallest ball, i.e. $\|X\| \leq r_{\min}(\Gamma)$.

The above assertion is easily proven by making use of Banach's Fixed Point Theorem (cf. [16]). Furthermore, it is then shown that if the value of $l = \pm 1$ is fixed, the solution X does not depend on a specific contour $\Gamma \subset D^l$ satisfying (3.6). Moreover, the bound on the norm of X may be optimized with respect to the admissible contours Γ^l in the form $\|X\| \leq r_0(B)$ with

$$r_0(B) := \inf_{\Gamma^l: \omega(B, \Gamma^l) > 0} r_{\min}(\Gamma^l), \quad (3.8)$$

where $\omega(B, \Gamma^l) = d^2(\Gamma^l) - 4\mathcal{V}_0(B, \Gamma^l)$. Unlike $r_0(B)$, the solution X depends on l , and thus we will supply its notation with the index l writing $X^{(l)}$. As it is seen from the next statement, the operators $Z^{(l)} = A_1 + X^{(l)}$, $l = \pm 1$, may be understood as operator roots of the continued Schur complement M_1 .

Two assertions below (Theorem 3.2 and Theorem 3.4) may be proven in exactly the same way as the corresponding statements in [16] (see [16, Theorems 4.1 and 4.9]; also cf. [17]), only the plus and minus signs interchange in certain places. Thus, we present these assertions without a proof.

Theorem 3.2. *Let Γ^l be an admissible contour satisfying (3.6) and let $Z^{(l)} = A_1 + X^{(l)}$ where $X^{(l)}$ is the corresponding unique solution of (3.4) mentioned in Theorem 3.1. Then, for $z \in \mathbb{C} \setminus \Gamma^l$, the Schur complement $M_1(z, \Gamma^l)$ admits the following factorization:*

$$M_1(z, \Gamma^l) = F_1(z, \Gamma^l) (Z^{(l)} - z), \quad (3.9)$$

where

$$F_1(z, \Gamma^l) = I + \int_{\Gamma^l} d\mu K'_B(\mu) (Z^{(l)} - \mu)^{-1} (\mu - z)^{-1} \quad (3.10)$$

is a bounded operator on \mathfrak{A}_1 . Moreover, if $\text{dist}(z, \sigma_1) \leq \frac{1}{2}d(\Gamma^l)$ then for sure $F_1(z, \Gamma^l)$ has a bounded inverse.

Following the Markus and Matsaev factorization result for holomorphic operator-valued functions [15] we interpret the factorization property (3.9) in the sense that $Z^{(l)}$ is the operator root of the analytically continued Schur complement $M_1(\cdot, \Gamma^l)$.

As an elementary consequence of Theorem 3.2, we obtain the following corollary.

Corollary 3.3. *The spectrum $\sigma(Z^{(l)})$ of the operator $Z^{(l)} = A_1 + X^{(l)}$ lies in the closed $r_0(B)$ -neighborhood $\mathcal{O}_{r_0}(\sigma_1)$ of the spectrum of A_1 in \mathbb{C} . Moreover, the spectrum of $M_1(\cdot, \Gamma^l)$ lying in the closed $d(\Gamma)/2$ -neighborhood $\mathcal{O}_{d(\Gamma)/2}(\sigma_1)$ of σ_1 in \mathbb{C} is nothing but the spectrum of $Z^{(l)}$.*

Let

$$\Omega^{(l)} = \int_{\Gamma^l} d\mu (Z^{(-l)*} - \mu)^{-1} K'_B(\mu) (Z^{(l)} - \mu)^{-1} \quad (3.11)$$

where Γ^l denotes an admissible contour satisfying the condition (3.6).

Theorem 3.4. *The operators $\Omega^{(l)}$, $l = \pm 1$, have the following properties (cf. [18]):*

$$\|\Omega^{(l)}\| < \frac{\mathcal{V}_0(B, \Gamma)}{\frac{1}{4}d^2(\Gamma)} < 1, \quad (3.12)$$

$$\Omega^{(-l)} = \Omega^{(l)*}, \quad (3.13)$$

$$-\frac{1}{2\pi i} \int_{\gamma} dz [M_1(z, \Gamma^l)]^{-1} = (I - \Omega^{(l)})^{-1}, \quad (3.14)$$

$$-\frac{1}{2\pi i} \int_{\gamma} dz z [M_1(z, \Gamma^l)]^{-1} = (I - \Omega^{(l)})^{-1} Z^{(-l)*} = Z^{(l)} (I - \Omega^{(l)})^{-1} \quad (3.15)$$

where γ stands for an arbitrary rectifiable closed contour going in the positive direction around the spectrum of $Z^{(l)}$ inside the set $\mathcal{O}_{d(\Gamma)/2}(\sigma_1)$.

The expressions (3.14) and (3.15) allow us, in principle, to find the operators $Z^{(l)}$ and, thus, to solve the equation (3.4) only by using the contour integration of the inverse of the continued Schur complement $[M_1(z, \Gamma^l)]^{-1}$. From 3.15) it also follows that the operators $Z^{(-1)*}$ and $Z^{(+1)}$ are similar to each other and, thus, the spectrum of $Z^{(-1)*}$ coincides with that of $Z^{(+1)}$.

4. SOLVABILITY OF THE OPERATOR RICCATI EQUATIONS IN CASE OF ABSENCE OF THE REAL SPECTRUM AND RESONANCES

To avoid some purely technical complications, in the rest of the paper we assume that the operator A_1 is bounded.

Recall that we work under the assumption that the spectrum of A_0 is absolutely continuous and that it coincides with the closure $\bar{\Delta}_0$ of the interval $\Delta_0 \subset \mathbb{R}$. The interval Δ_0 is a part of the boundary of the continuation domain $D^l \subset \mathbb{C}^l$, $l = \pm 1$, for the Schur complement M_1 . Suppose it so happened that the spectrum of the operator $Z^{(l)}$ is separated from the spectrum of A_0 and that, in addition, the (complex) spectrum of $Z^{(l)}$ in the domain D^l is empty. In other words, let us make the following assumption.

Hypothesis 4.1. *Let the (self-adjoint) operator A_1 be bounded. Assume the hypothesis of Theorem 3.1 for an admissible contour $\Gamma^l \subset D^l$, $l = \pm 1$ and let $Z^{(l)} = A_1 + X^{(l)}$ where $X^{(l)}$ denotes the corresponding unique solution to (3.6). Suppose that*

$$\text{dist}(\text{spec}(Z^{(l)}), D^l) > 0. \quad (4.1)$$

Remark 4.2. One notices that under Hypothesis 4.1 and, in particular, under the assumption (4.1) the resolvent $(Z^{(l)} - \mu)^{-1}$ is uniformly bounded on $\bar{\Delta}_0$, i.e. $\sup_{\mu \in \bar{\Delta}_0} \|(Z^{(l)} - \mu)^{-1}\| < \infty$, and hence

$$\int_{\Delta_0} d\mu \|K'_B(\mu)\| \|(Z^{(l)} - \mu)^{-1}\|^2 < \infty. \quad (4.2)$$

Under Hypothesis 4.1 one may talk on the existence of solutions to the operator Riccati equations (1.9) and (1.8).

Lemma 4.3. *Assume Hypothesis 4.1 for some $l = \pm 1$ and set*

$$Y^{(l)} = \int_{\Delta_0} E_0(d\mu) B(Z^{(l)} - \mu)^{-1}. \quad (4.3)$$

The operator $Y^{(l)}$ is a bounded operator from \mathfrak{A}_1 to \mathfrak{A}_0 ,

$$\|Y^{(l)}\| \leq \left(\int_{\Delta_0} d\mu \|K'_B(\mu)\| \|(Z^{(l)} - \mu)^{-1}\|^2 \right)^{1/2}. \quad (4.4)$$

Moreover, $Y^{(l)}$ is a strong solution to the operator Riccati equation (1.8).

Proof. By the hypothesis, there is no spectrum of $Z^{(l)}$ in D^l and, hence, on the closure of the subdomain $D(\Gamma^l)$ bounded by the interval Δ_0 and the curve Γ^l . In particular, $\text{dist}(\Delta_0, \text{spec}(Z^{(l)})) > 0$ and one can transform the integration contour Γ^l into Δ_0 and equivalently replace equation (3.3) by the equation

$$Z^{(l)} = A_1 - B^* \int_{\Delta_0} E_0(d\mu) B(Z^{(l)} - \mu)^{-1}. \quad (4.5)$$

The integrals on the right-hand sides of (4.3) and (4.5) are understood in the strong sense. Clearly, due to (2.2) and (4.2) we have

$$\begin{aligned} \left\| \int_{\Delta_0} E_0(d\mu) B(Z^{(l)} - \mu)^{-1} x \right\|^2 &= \left\langle \int_{\Delta_0} B^* E_0(d\mu) B(Z^{(l)} - \mu)^{-1} x, (Z^{(l)} - \mu)^{-1} x \right\rangle \\ &= \left\langle \int_{\Delta_0} d\mu K'_B(\mu) (Z^{(l)} - \mu)^{-1} x, (Z^{(l)} - \mu)^{-1} x \right\rangle \\ &\leq \int_{\Delta_0} d\mu \|K'_B(\mu)\| \|(Z^{(l)} - \mu)^{-1}\|^2 \|x\|^2, \end{aligned} \quad (4.6)$$

which implies (4.4). From (4.5) it also follows that

$$Z^{(l)} = A_1 - B^* Y^{(l)}, \quad (4.7)$$

and by (4.3) the operator $Y^{(l)}$ itself satisfies the equation

$$Y^{(l)} = \int_{\Delta_0} E_0(d\mu) B(A_1 - B^* Y^{(l)} - \mu)^{-1}. \quad (4.8)$$

Then [3, Theorem 5.5] (also cf. [2, Theorem 3.4]) applies and one concludes that $Y^{(l)}$ is a strong solution to (1.8), completing the proof. \square

Corollary 4.4. *The operator $\widehat{Y}^{(l)} = Y^{(l)*}$ is a strong solution to the operator Riccati equation (1.9). The graphs*

$$\mathfrak{L}_0^{(l)} = \mathcal{G}(\widehat{Y}^{(l)}) \text{ and } \mathfrak{L}_1^{(l)} = \mathcal{G}(Y^{(l)}) \quad (4.9)$$

are mutually J -orthogonal invariant subspaces of the block operator matrix L . That is,

$$\langle Jx, y \rangle = 0 \text{ for any } x \in \mathcal{G}(\widehat{Y}^{(l)}), y \in \mathcal{G}(Y^{(l)}) \quad (4.10)$$

and $Lx \in \mathcal{G}(\widehat{Y}^{(l)})$ whenever $x \in \text{Dom}(L) \cap \mathcal{G}(\widehat{Y}^{(l)})$ and $Ly \in \mathcal{G}(Y^{(l)})$ whenever $y \in \text{Dom}(L) \cap \mathcal{G}(Y^{(l)})$.

Remark 4.5. The following inequality holds:

$$\|Y^{(l)}\| \geq 1 \quad (\text{and, hence, } \|\widehat{Y}^{(l)}\| = \|Y^{(l)}\| \geq 1), \quad l = \pm 1. \quad (4.11)$$

Proof. Suppose the opposite, that is, $Y^{(l)} < 1$. Then by [4, Theorem 5.3] (see also [1, Theorem 4.1] and [18, Theorem 3.2]) the operator matrix L is similar to a self-adjoint operator and, hence, the spectrum of L is purely real. Moreover, in such a case the spectrum L is given by

$$\text{spec}(L) = \text{spec}(Z^{(l)}) \cup \text{spec}(\widehat{Z}^{(l)}), \quad (4.12)$$

where

$$\widehat{Z}^{(l)} := A_0 + B\widehat{Y}^{(l)} = A_0 + BY^{(l)*}, \quad \text{Dom}(\widehat{Z}^{(l)}) = \text{Dom}(A_0). \quad (4.13)$$

Thus, in particular, $\text{spec}(Z^{(l)}) \subset \mathbb{R}$. But this is not the case since Corollary 3.3 and Hypothesis 4.1 imply the inclusion $\text{spec}(Z^{(l)}) \subset D^{(-l)}$, i.e. the set $\text{spec}(Z^{(l)})$ is purely complex. This contradiction completes the proof. \square

Thus, under Hypothesis 4.1 the angular operators $Y^{(l)}$, $l = \pm 1$, are definitely not strict contractions, and it is possible that 1 is eigenvalue for $Y^{(l)*}Y^{(l)}$ and, hence, for $\widehat{Y}^{(l)*}\widehat{Y}^{(l)} = Y^{(l)}Y^{(l)*}$. In this case the invariant graph subspaces (4.9) of L have a non-trivial intersection, $\mathfrak{L}_0^{(l)} \cap \mathfrak{L}_1^{(l)} \neq \{0\}$. The equality $\mathfrak{L}_0^{(l)} \cap \mathfrak{L}_1^{(l)} = \{0\}$ and the linear independence of the invariant graph subspaces (4.9) is ensured provided it is known that

$$1 \notin \text{spec}(Y^{(l)*}Y^{(l)}) \text{ (and, hence, } 1 \notin \text{spec}(Y^{(l)}Y^{(l)*}) \text{)}. \quad (4.14)$$

In the latter case we would have two versions of the direct decomposition (see, e.g., [4, Lemma 2.6 and Remark 2.7])

$$\mathfrak{H} = \mathfrak{L}_0^{(l)} \dot{+} \mathfrak{L}_1^{(l)}, \quad l = \pm 1. \quad (4.15)$$

Furthermore, with respect to the decomposition (4.15) the operator L would read as the block diagonal matrix (see, e.g., [4, Corollary 2.9])

$$Z = \text{diag}(\widehat{Z}^{(l)}, Z^{(l)}), \quad l = \pm 1. \quad (4.16)$$

What is a criterion for the situation (4.14) to take place is an open problem. In the explicitly solvable example discussed below in Section 6 we have just the opposite situation: 1 is an eigenvalue of both $Y^{(+)*}Y^{(+)}$ and $Y^{(-)*}Y^{(-)}$.

5. FESHBACH CASE

In the present section we consider the spectral situation that resembles the Feshbach one in the case of self-adjoint block operator matrices. Namely, we assume that the spectrum of the self-adjoint operator A_1 only consists of isolated eigenvalues of finite multiplicities and all these eigenvalues are embedded into the absolutely continuous spectrum of the self-adjoint operator A_0 . If A_1 is bounded then the Hilbert space \mathfrak{A}_1 is necessarily finite-dimensional and A_1 is finite rank.

We start the discussion of this case with a remark that the operator-valued function $K_B(\mu)$ for $\mu \in \mathbb{R}$ is non-decreasing. Hence, under our assumptions on analytic properties of K_B , the derivative $K'_B(\mu)$ is a non-negative operator on Δ_0 ,

$$\langle K'_B(\mu)x, x \rangle \geq 0, \text{ for any } \mu \in \Delta_0 \text{ and any } x \in \mathfrak{A}_1. \quad (5.1)$$

To simply future references, we adopt the following hypothesis.

Hypothesis 5.1. Let $\dim(\mathfrak{A}_1) < \infty$ (and, hence, the spectrum σ_1 of the operator A_1 consists only of a finite number of isolated eigenvalues of finite multiplicities). Suppose that $\sigma_1 \subset \Delta_0$ and that there is an admissible contour $\Gamma \subset D^\pm$ such that $\mathcal{V}_0(B, \Gamma) < \frac{1}{4}d^2(\Gamma)$. Furthermore, assume that the derivative $K'_B(\mu)$ is strictly positive and uniformly bounded from below on the intersection $\Delta_0 \cap \mathcal{O}_{r_0}(\sigma_1)$ of the interval Δ_0 with the r_0 -neighborhood of the spectrum of A_1 , where $r_0 \equiv r_0(B) > 0$ is given by (3.8). That is, assume there is $c_0 > 0$ such that

$$\langle K'_B(\mu)x, x \rangle \geq c_0 \|x\|^2 \quad \text{for any } \mu \in \Delta_0 \cap \mathcal{O}_{r_0}(\sigma_1) \text{ and any } x \in \mathfrak{A}_1. \quad (5.2)$$

Notice that the function $W_1(z)$ given by (2.1) is Herglotz. One easily verifies that the following limiting equalities hold:

$$\operatorname{Im} \langle W_1(\lambda \pm i0)x, x \rangle := \lim_{\varepsilon \downarrow 0} \langle W_1(\lambda \pm i\varepsilon)x, x \rangle = \pm \pi \langle K'_B(\lambda)x, x \rangle, \quad (5.3)$$

for any $\lambda \in \Delta_0$ and any $x \in \mathfrak{A}_1$.

Lemma 5.2. Assume Hypothesis 5.1 and let Γ^l , $l = \pm 1$, be an admissible contour satisfying condition $\mathcal{V}_0(B, \Gamma^l) < \frac{1}{4}d^2(\Gamma^l)$. Let $X^{(l)}$ be the corresponding unique solution of (3.4) mentioned in Theorem 3.1 and set $Z^{(l)} = A_1 + X^{(l)}$. The spectrum of $Z^{(l)}$ only consists of finite number of isolated eigenvalues of finite (algebraic) multiplicities and none of these eigenvalues is real.

Proof. Since the space \mathfrak{A}_1 is finite-dimensional, the spectrum of $Z^{(l)}$ is automatically formed only of isolated eigenvalues with finite algebraic multiplicities and the number of these eigenvalues is finite.

Suppose that $u_1 \in \mathfrak{A}_1$, $\|u_1\| = 1$, is an eigenvector of $Z^{(l)}$, $l = \pm 1$, corresponding to an eigenvalue z , i.e. $Z^{(l)}u_1 = zu_1$. From (3.5) it follows that

$$z = \langle A_1 u_1, u_1 \rangle + \langle W_1(z, \Gamma^l) u_1, u_1 \rangle. \quad (5.4)$$

One proves that $\operatorname{Im} z \neq 0$ by contradiction. Indeed, assume the opposite, i.e. that $z = \lambda \in \mathbb{R}$. Clearly, we have

$$W_1(\zeta, \Gamma^l) = W_1(\zeta) \quad \text{whenever } \zeta \in \mathbb{C}^{-l}, \quad l = \pm 1. \quad (5.5)$$

Combining (5.5) with (5.3) one observes that

$$\operatorname{Im} \langle W_1(\lambda, \Gamma^l) u_1, u_1 \rangle = \mp l \pi \langle K'_B(\lambda) u_1, u_1 \rangle \quad \text{for any } \lambda \in \Delta_0. \quad (5.6)$$

In view of Corollary 3.3, from (5.2) and (5.6) it follows that

$$|\operatorname{Im} \langle W_1(\lambda, \Gamma^l) u_1, u_1 \rangle| \geq \pi c_0 > 0. \quad (5.7)$$

Together with $\langle A_1 u_1, u_1 \rangle \in \mathbb{R}$ this means that for $z = \lambda \in \mathbb{R}$ the equality (5.4) is impossible. Thus one concludes that $Z^{(l)}$ has only non-real eigenvalues, completing the proof. \square

Remark 5.3. Given $l = \pm 1$, there is an open neighborhood of the interval $(\min \sigma_1, \max \sigma_1)$ in \mathbb{C} that contains no spectrum of $Z^{(l)}$. This follows from the fact that, by the continuity argument, at some complex neighborhood of the set $\mathcal{O}_{r_0}(\sigma_1) \cap \Delta_0$ the imaginary part of $W_1(z, \Gamma_1)$ should remain uniformly definite, keeping the same respective sign that it had on $\mathcal{O}_{r_0}(\sigma_1) \cap \Delta_0$.

In the Feshbach case under consideration, the spectrum of $Z^{(l)}$ represents a part of the “usual” spectrum of the block operator matrix L . In other words, unlike in the case of self-adjoint off-diagonal V in [16, 17], the spectrum of $Z^{(l)}$ contains no resonances. This is established in the following lemma.

Lemma 5.4. *Under Hypothesis 5.1, the operator $Z^{(l)}$, $l = \pm 1$, has no spectrum in the corresponding complex domain D^l and, thus, all the eigenvalues of $Z^{(l)}$ are simultaneously eigenvalues of the original (not yet continued) Schur complement $M_1(\cdot)$ and, hence, the eigenvalues of L .*

Proof. Consider the path of J -self-adjoint operators $L_t = A + tV$, $t \in [0, 1]$, $\text{Dom}(L_t) = \text{Dom}(A)$. With this path we associate the corresponding path of the (unique) solutions $X_t^{(l)}$ (from Theorem 3.1) to the respective transformator equations

$$X = t^2 W_1(A_1 + X, \Gamma^l). \quad (5.8)$$

as well as the path $Z_t^{(l)} = A_1 + X_t^{(l)}$, $t \in [0, 1]$. Obviously, we have $r_0(tB) \rightarrow 0$ as $t \rightarrow 0$ where the radius r_0 is given by (3.8). Using the same reasoning as in the proof of Lemma 5.2, by Corollary (3.3) we then conclude that $\text{spec}(Z_t^{(l)})$ lies in \mathbb{C}^{-l} and hence

$$\text{spec}(Z_t^{(l)}) \cap D^l = \emptyset \quad (5.9)$$

at least for sufficiently small t . Furthermore, by Remark 5.3 we have the strict separation of the spectrum of $Z_t^{(l)}$ from the real axis at every $0 < t \leq 1$. The solution $X_t^{(l)}$ is a real analytic and, hence, norm continuous in $t \in [0, 1]$. For varying t , we then apply the result on the continuity of finite systems of eigenvalues (see [13, Section IV.5]) to conclude that the eigenvalues of $Z_t^{(l)}$ can not jump from \mathbb{C}^{-l} to D^l . \square

6. THE SIMPLEST EXAMPLE

In this section we consider the operator matrix L of the form (1.1) where A_0 is the operator of multiplication by independent variable,

$$(A_0 u_0)(\mu) = \mu u_0(\mu), \quad (6.1)$$

on $\mathfrak{A}_0 = L_2(-\alpha, \alpha)$, $0 < \alpha < +\infty$. The spectrum of A_0 is absolutely continuous and fills the interval $[-\alpha, \alpha]$. We assume that $\mathfrak{A}_1 = \mathbb{C}$ and, thus, A_1 is the multiplication by a real number a_1

$$A_1 u_1 = a_1 u_1, \quad u_1 \in \mathbb{C}. \quad (6.2)$$

The latter should lie inside the continuous spectrum of A_1 , i.e. $a_1 \in (-\alpha, \alpha)$. The coupling operator $B : \mathfrak{A}_1 \rightarrow \mathfrak{A}_0$ is assumed to be the multiplication by another constant $b \geq 0$, namely

$$(B u_1)(\mu) := b u_1, \quad \mu \in [-\alpha, \alpha] \quad (6.3)$$

(that is, $B u_1$ is constant function on $[-\alpha, \alpha]$). Obviously, the adjoint operator B^* is given by

$$B^* u_0 = b \int_{-\alpha}^{\alpha} u_0(\mu) d\mu. \quad (6.4)$$

The self-adjoint analog of such an operator matrix L (with the lower left entry in (1.1) replaced by simply B^*) as well as somewhat more complex self-adjoint operator matrices have been discussed in detail in [16, Section 8]. Notice that the self-adjoint analog of L represents a particular case of one of the celebrated Friedrichs models [10].

The spectral function $E^0(\mu)$, $\mu \in \mathbb{R}$, of the multiplication operator A_0 is given by (see, e. g., [8])

$$(E^0(\mu) u_0)(v) = \chi_{(-\infty, \mu) \cap [-\alpha, \alpha]}(v) u_0(v), \quad u_0 \in \mathfrak{A}_0, \quad v \in [-\alpha, \alpha], \quad (6.5)$$

where $\chi_\delta(\cdot)$ denotes the characteristic function (indicator) of a Borel set $\delta \subset \mathbb{R}$; $\chi_\delta(\mu) = 1$ if $\mu \in \delta$ and $\chi_\delta(\mu) = 0$ if $\mu \in \mathbb{R} \setminus \delta$. Hence, the product (2.2) for $-\alpha \leq \mu \leq \alpha$ reads

$$K_B(\mu) = b^2 \int_{-\alpha}^{\mu} dv = b^2(\mu + \alpha),$$

and the derivative $K'_B(\mu)$, $\mu \in (-\alpha, \alpha)$, is simply the constant, $K'_B(\mu) = b^2$, admitting analytic continuation anywhere on the complex plane. Thus, in the case under consideration one can choose the whole half-plane \mathbb{C}^+ as D^+ and the whole half-plane \mathbb{C}^- as D^- .

The Schur complement (1.12) reads

$$M_1(z) = a_1 - z + b^2 \int_{-\alpha}^{\alpha} \frac{d\mu}{\mu - z} \quad \text{for } z \in \mathbb{C} \setminus [-\alpha, \alpha], \quad (6.6)$$

while its values $M_1(\lambda \pm i0)$, $\lambda \in (-\alpha, \alpha)$, on the rims of the cut are defined as the respective limits in $z = \lambda \pm i\varepsilon$ as $\varepsilon \searrow 0$. The corresponding continuations (2.5) are given by

$$M_1(z, \Gamma^l) = a_1 - z + b^2 \int_{\Gamma^l} \frac{d\mu}{\mu - z}, \quad z \in \mathbb{C} \setminus \Gamma^l, \quad l = \pm 1. \quad (6.7)$$

In this case the basic equation (3.3) coincides with the equation $M_1(z, \Gamma^\pm) = 0$ and the solutions $Z^{(\pm)}$ if they exist are simply the numbers, $Z^{(\pm)} = z^{(\pm)} \in \mathbb{C}$. One easily verifies by inspection that the function $M_1(z)$ does not have real roots $z \in \mathbb{R} \setminus [-\alpha, \alpha]$ and, surely, the same holds true for the functions (6.7). Furthermore, for $b > 0$ none of the functions (6.6) and (6.7) has roots in $(-\alpha, \alpha)$ since

$$\text{Im} M_1(\lambda, \Gamma^\pm) = \text{Im} M_1(\lambda \mp i0) = \mp \pi b^2 \quad \text{whenever } \lambda \in (-\alpha, \alpha).$$

Finally, one notices that $z = -\alpha$ and $z = \alpha$ are branching points for the Riemann surface of $M_1(\cdot)$.

For the remaining part of the section we set

$$a_1 = 0 \quad (6.8)$$

and as Γ^\pm take semi-circumferences, $\Gamma^\pm = \{z : |z| = \alpha, z \in \mathbb{C}^\pm\}$. Then, obviously, $d := d(\Gamma^\pm) = \alpha$ and $\mathcal{V}_0(B, \Gamma^\pm) = \pi b^2 \alpha$. The condition (3.6) acquires the form

$$b^2 < \frac{\alpha}{4\pi} \quad (6.9)$$

and, given the sign $l = \pm 1$, Theorem 3.1 ensures the existence of a unique solution $Z^{(l)} = z^{(l)}$ to the basic equation (3.3) in the open circle $|z| < r_{\max}$, $z \in \mathbb{C}$, where

$$r_{\max} = d - \sqrt{\mathcal{V}_0(B, \Gamma^\pm)} = \alpha - \sqrt{\pi b^2 \alpha} > \frac{\alpha}{2}. \quad (6.10)$$

Theorem 3.1 also guarantees that, in fact, the unique solution $Z^{(l)} = z^{(l)}$ belongs to the smaller circle $|z| \leq r_{\min}$, $z \in \mathbb{C}$, where

$$r_{\min} = \frac{d}{2} - \sqrt{\frac{d^2}{4} - \mathcal{V}_0(B, \Gamma^\pm)} = \frac{\pi b^2}{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\pi b^2}{\alpha}}} < \frac{\alpha}{2}. \quad (6.11)$$

By Lemmas 5.2 and 5.4 one then concludes that $\text{Im} z^{(+)} < 0$ and $\text{Im} z^{(-)} > 0$ and, thus, that z^\pm are the roots of the original (not continued) Schur complement $M_1(\cdot)$ given by (6.6). Under the

assumption (6.8), an elementary inspection shows that, for the function $M_1(\cdot)$, the roots $z^{(\pm)}$ are purely imaginary,

$$z^{(-)} = -z^{(+)} = iy \quad \text{with } y > 0. \quad (6.12)$$

Moreover, each of these two roots is unique for the upper and lower half-planes \mathbb{C}^+ and \mathbb{C}^- , respectively. Obviously, the equation $M_1(z^{(\pm)}) = 0$ with $M_1(\cdot)$ given by (6.6) for $a_1 = 0$ and $b > 0$ is equivalent to

$$1 = b^2 \int_{-\alpha}^{\alpha} \frac{d\mu}{\mu^2 + y^2}, \quad y > 0. \quad (6.13)$$

Evaluating the integral in (6.13) one finds that y is the unique positive solution to the equation

$$y = 2b^2 \arctan \frac{\alpha}{y} \quad (6.14)$$

and for the existence of this solution no smallness requirement like (6.9) is needed: In fact, the unique positive solution to (6.14) and, hence, the corresponding unique roots (6.12) to the Schur complement (6.6) exist for any $\alpha > 0$ and $b^2 > 0$.

According to (4.3) and (6.5), for the angular operators $Y^{(l)}$, $l = \pm 1$, associated with the above two solutions $Z^{(l)} = z^{(l)}$ we have

$$(Y^{(\pm)}u_1)(\mu) = \frac{b}{z^{(\pm)} - \mu}u_1 = -\frac{b}{\mu \pm iy}u_1 \quad \text{for any } \mu \in [-\alpha, \alpha] \text{ and } u_1 \in \mathfrak{A}_1 = \mathbb{C}, \quad (6.15)$$

where y is the unique positive solution of (6.14). Obviously, from (6.15) for the norm of $Y^{(l)}$ one infers

$$\|Y^{(l)}\| = b \left(\int_{-\alpha}^{\alpha} \frac{d\mu}{\mu^2 + y^2} \right)^{1/2} = 1, \quad l = \pm 1, \quad (6.16)$$

where the equality (6.13) has been taken into account at the last step.

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